Journal of Approximation Theory **94**, 306–329 (1998)
Article No. AT983187

# Generalization of the Left Bernstein Quasi-Interpolants

Yasuo Kageyama

*Department of Information Systems Engineering, Kobe University of Mercantile Marine, 5-1-1, Fukae-Minami-machi, Higashinada-ku, Kobe, 658-0022, Japan*
E-mail: kage@cc.kshosen.ac.jp

*Communicated by Dany Leviatan*

Received November 25, 1996; accepted in revised form June 4, 1997

P. Sablonnière introduced the so-called left Bernstein quasi-interpolant, and proved that the sequence of the approximating polynomials converges pointwise in high-order rate to each sufficiently smooth approximated function. On the other hand, Z.-C. Wu proved that the sequence of the norms of the operators is bounded. In this paper, we extract the essence why Sablonnière's operator exhibits good convergence and stability properties, and we clarify a sufficient condition for general operators to have similar properties. Moreover, regarding the family of the general operators, we derive detailed results about the derivatives of the approximating polynomials that estimate their uniform convergence degree, using a convenient differentiability condition on approximated functions. Our results readily imply all the preceding ones. 

## 1. INTRODUCTION

The Bernstein operator $B_n$ of order $n \in \mathbf{N}$ is defined as

$$B_n f(x) = \sum_{v=0}^{n} f\left(\frac{v}{n}\right)\binom{n}{v} x^v (1-x)^{n-v} \qquad (f\colon [0,1] \to \mathbf{R},\ x \in [0,1]),$$

while the Lagrange (interpolation) operator $L_n$ of the same nodes as $B_n$ is represented as

$$L_n f(x) = \sum_{v=0}^{n} f\left(\frac{v}{n}\right)\binom{nx}{v}\binom{n(1-x)}{n-v} \qquad (f\colon [0,1] \to \mathbf{R},\ x \in [0,1]).$$

There are many classical results on the Bernstein operator [1, 2]. In this paper, we particularly notice P. Sablonnière's work [5, 6]. He defined the *left Bernstein quasi-interpolant* operator $B_n^{(K)}$ as

$$B_n^{(K)} f = \sum_{k=0}^{K} \alpha_k^n (B_n f)^{(k)} \qquad (f\colon [0,1] \to \mathbf{R}),$$



0021-9045/98 $25.00

where $K$ is an integer satisfying $0 \leqslant K \leqslant n$ and $\alpha_k^n$ are polynomials of degree at most $k$ satisfying

$$L_n f = \sum_{k=0}^{n} \alpha_k^n (B_n f)^{(k)} \qquad (f\colon [0, 1] \to \mathbf{R});$$

and he proved in [6] that

$$\lim_{n\to\infty} n^{l+1}(B_n^{(2l)} f(x) - f(x)) = \frac{(-1)^l X^l (4l(l+1)(1-2x)\, f^{(2l+1)}(x) + 3X f^{(2l+2)}(x))}{3 \cdot 2^{l+1}(l+1)!},$$

$$\lim_{n\to\infty} n^{l+1}(B_n^{(2l+1)} f(x) - f(x)) = \frac{(-1)^l X^{l+1} f^{(2l+2)}(x)}{2^{l+1}(l+1)!},$$

where $l$ is a non-negative integer, $f \in C^{2l+3}[0, 1]$, $x \in [0, 1]$, and $X$ denotes $x(1-x)$. Moreover, Z.-C. Wu proved in [7] that the sequence $\{\|B_n^{(K)}\|\}_{n=K}^{\infty}$ is bounded for each $K$, where $\|\cdot\|$ is the operator norm subordinate to the uniform norm on $C[0, 1]$.

The aim of this paper is to extract the essence of the above-mentioned facts on $B_n^{(K)}$, to clarify the structure of general operators that have similar properties to those of Sablonnière's operator, and to derive more general and more detailed results than the preceding ones, which imply their theorems as a part of a "corollary."

Throughout the paper, we adopt the following notations and conventions:

- $\sum_{k=p}^{q} \cdot = 0$, $\prod_{k=p}^{q} \cdot = 1$ if $p, q \in \mathbf{Z}$, $q < p$;
- the symbol $\sum_{k=P}^{Q}$ stands for $\sum_{k=\max P}^{\min Q}$ if $P, Q$ are finite sets of integers;
- the symbol $\mathbf{N}_0$ denotes $\mathbf{N} \cup \{0\}$;
- the symbol $a^{(n)}$ stands for $\prod_{k=0}^{n-1} (a-k)$ and $\binom{a}{n} = a^{(n)}/n!$ if $a \in \mathbf{R}$, $n \in \mathbf{N}_0$;
- the symbol $n!!$ stands for $\prod_{k=0}^{[(n-1)/2]} (n-2k)$ if $n \in \mathbf{Z}$, $n \geqslant -1$;
- the symbol $f^{[n]}$ stands for $f^{(n)}/n!$ if $f$ is a function and $n \in \mathbf{N}_0$;
- the symbol $\mathbf{P}_n$ denotes the set of polynomials of degree at most $n \in \mathbf{N}_0$ with real coefficients;
- the symbol $X$ denotes $x(1-x)$;
- the symbol $e_n$ denotes the polynomial of degree $n$ defined as $e_n(x) = (1-2x)^{n-2[n/2]} X^{[n/2]}$ for every $n \in \mathbf{N}_0$, i.e., $e_{2m}(x) = X^m$, $e_{2m+1}(x) = (1-2x) X^m$ for every $m \in \mathbf{N}_0$;

- the symbol $\|\cdot\|$ denotes the uniform functional norm on $C[0, 1]$ or the operator norm subordinate to it;
- the symbol $\Delta_h$ denotes the forward difference operator of stepsize $h$ ($h \in \mathbf{R}$, $h > 0$).

## 2. MAIN RESULTS

Our main results are summed up in the following four theorems, whose kernel is Theorem 2.4.

THEOREM 2.1. *Let $n \in \mathbf{N}$ and $T$ be an operator on $\{f \mid f\colon [0, 1] \to \mathbf{R}\}$. Then the following two conditions are equivalent*:

(1) *$T$ is represented as the form $Tf = \sum_{\nu=0}^{n} f(\nu/n)\, \tau_\nu$ ($\tau_\nu \in \mathbf{P}_n$, $f\colon [0, 1] \to \mathbf{R}$) and $T\mathbf{P}_m \subseteq \mathbf{P}_m$ ($0 \leqslant m \leqslant n$);*

(2) *there exist unique $V_k \in \mathbf{P}_k$ ($0 \leqslant k \leqslant n$) such that*

$$Tf = \sum_{k=0}^{n} V_k (B_n f)^{[k]} \qquad (f\colon [0, 1] \to \mathbf{R}),$$

*where $V_{k,l}$ ($k, l \in \mathbf{N}_0$, $k \leqslant n$, $l \leqslant n-k$) are determined by the following recursion formula and $V_k$ can be identified with $V_{k,0}$:*

$$\begin{cases} V_{-1,l} = 0 \qquad (0 \leqslant l \leqslant n-1), \\ V_{0,l}(x) = T(\cdot - x)^l\,(x) = \sum_{\nu=0}^{n} (\nu/n - x)^l\, \tau_\nu(x) \qquad (x \in [0, 1])\ (0 \leqslant l \leqslant n), \\ (n-k)\, V_{k+1,l} = nV_{k,l+1} - k(e_1 V_{k,l} + e_2 V_{k-1,l}) \\ \qquad (0 \leqslant k \leqslant n-1,\ 0 \leqslant l \leqslant n-k-1). \end{cases}$$

THEOREM 2.2. *For each $n \in \mathbf{N}$, there exist unique $U_k^n \in \mathbf{P}_k$ ($0 \leqslant k \leqslant n$) such that*

$$L_n f = \sum_{k=0}^{n} U_k^n (B_n f)^{[k]} \qquad (f\colon [0, 1] \to \mathbf{R}),$$

*where $U_k^n$ are determined by the recursion formula*

$$\begin{cases} U_{-1}^n = 0, \qquad U_0^n = 1, \\ (n-k)\, U_{k+1}^n = -k(e_1 U_k^n + e_2 U_{k-1}^n) \qquad (0 \leqslant k \leqslant n-1). \end{cases}$$

*Remark.* We use this notation $U_k^n$ throughout the paper.

THEOREM 2.3. *For each $n \in \mathbf{N}$, $k \in \mathbf{N}_0$ $(k \leqslant n)$, we expand $U_k^n$ as the form*

$$U_k^n = \sum_{l=0}^{k} u_{k,l}(n)\, e_l.$$

*Then the coefficients are estimated asymptotically as follows for every $k, l \in \mathbf{N}_0$:*

$$\begin{cases} u_{2k,2l+1}(n) = 0 & \text{for all} \quad n \geqslant 2k \quad \text{if} \quad l \leqslant k-1, \\ u_{2k,2l}(n) = O(n^{l-2k}) & (n \to \infty) \quad \text{if} \quad l \leqslant k-1, \\ \lim_{n \to \infty} n^k u_{2k,2k}(n) = (-1)^k (2k-1)!!; \end{cases}$$

$$\begin{cases} u_{2k+1,2l}(n) = 0 & \text{for all} \quad n \geqslant 2k+1 \quad \text{if} \quad l \leqslant k, \\ u_{2k+1,2l+1}(n) = O(n^{l-2k-1}) & (n \to \infty) \quad \text{if} \quad l \leqslant k-1, \\ \lim_{n \to \infty} n^{k+1} u_{2k+1,2k+1}(n) = \tfrac{2}{3}(-1)^{k+1} k(2k+1)!!. \end{cases}$$

*Accordingly, they are roughly estimated as*

$$u_{k,l}(n) = O(n^{[l/2]-k}) \qquad (n \to \infty) \quad \text{for every } k, l \in \mathbf{N}_0 \;\; (l \leqslant k).$$

*In addition,*

$$\| U_k^n \| = O(n^{[k/2]-k}) \qquad (n \to \infty) \quad \text{for every } k \in \mathbf{N}_0.$$

*Remark.* We use this notation $u_{k,l}(n)$ throughout the paper.

THEOREM 2.4. *Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators on $\{f \mid f\colon [0,1] \to \mathbf{R}\}$ such that for each $n \in \mathbf{N}$, $T_n$ is represented as the form $T_n f = \sum_{v=0}^{n} f(v/n)\, \tau_{n,v}$ $(\tau_{n,v} \in \mathbf{P}_n,\; f\colon [0,1] \to \mathbf{R})$ and $T_n \mathbf{P}_m \subseteq \mathbf{P}_m$ $(0 \leqslant m \leqslant n)$. According to Theorem 2.1, we expand*

$$T_n f = \sum_{k=0}^{n} V_k^n (B_n f)^{[k]} \qquad (f\colon [0,1] \to \mathbf{R})$$

*and furthermore*

$$V_k^n = \sum_{l=0}^{k} v_{k,l}(n)\, e_l.$$

*Let $\alpha \in \mathbf{N}_0$ and suppose there exists a $K \in \mathbf{N}_0$ $(K \geqslant 2\alpha)$ such that for every $k, l \in \mathbf{N}_0$ the following conditions are satisfied:*

(a) $V_k^n = 0$ $(K < k \leqslant n)$ *for all* $n > K$;

(b) $v_{k,l}(n) = O(n^{[l/2]-k})$ $(n \to \infty)$ *if* $l \leqslant k \leqslant K$;

(c) $\| V_k^n - U_k^n \| = o(n^{-\alpha})$ $(n \to \infty)$ *if* $k \leqslant K$.

*Then* $\{T_n\}_{n=1}^{\infty}$ *has the following properties*:

(1) *for all* $p, q, r \in \mathbf{N}_0$, *there exists a constant* $M$ *such that for all* $n \in \mathbf{N}$ *and for all* $f \in C^r[0, 1]$

$$\| e_{2p} (T_n f)^{(q+r)} \| \leqslant M n^{q - \min\{p, [q/2]\}} \| f^{(r)} \|;$$

(2) *for all* $\beta, \gamma \in \mathbf{N}_0$ $(\beta \leqslant \alpha)$ *and for all* $f \in C^{2\beta+\gamma}[0, 1]$,

$$\| (T_n f)^{(\gamma)} - f^{(\gamma)} \| = o(n^{-\beta}) \qquad (n \to \infty);$$

(3) *if* $\lim_{n \to \infty} n^{\alpha+1} (V_k^n - U_k^n) = R_k$ *in the sense of* $\| \cdot \|$ $(0 \leqslant k \leqslant 2\alpha + 2)$, *then for all* $\gamma \in \mathbf{N}_0$ *and for all* $f \in C^{2\alpha+\gamma+2}[0, 1]$,

$$\lim_{n \to \infty} n^{\alpha+1} ((T_n f)^{(\gamma)} - f^{(\gamma)}) = \left( \sum_{k=0}^{2\alpha+2} R_k f^{[k]} \right)^{(\gamma)} \qquad \textit{in the sense of } \| \cdot \|.$$

Proofs of these theorems will be given in the later sections.

Recall $\alpha_k^n$ in Section 1 and note that $\alpha_k^n = U_k^n / k!$ because Theorem 2.2 guarantees the uniqueness of $U_k^n$. Though they were given in [5–7] by very complicated recurrence relations, now we can calculate them from a simple three-term recursion formula.

The left Bernstein quasi-interpolant operators $B_n^{(K)}$ were not defined when $n < K$, however, now we can redefine them for all $K \in \mathbf{N}_0$ and for all $n \in \mathbf{N}$ as

$$B_n^{(K)} f = \sum_{k=0}^{\{K, n\}} U_k^n (B_n f)^{[k]} \qquad (f\colon [0, 1] \to \mathbf{R}).$$

The above theorems imply the following corollary regarding $B_n^{(K)}$.

COROLLARY 2.1. *Let* $K \in \mathbf{N}_0$ *and* $\alpha = [K/2]$. *Then* $\{B_n^{(K)}\}_{n=1}^{\infty}$ *has the following properties*:

(1) *for all* $p, q, r \in \mathbf{N}_0$, *there exists a constant* $M$ *such that for all* $n \in \mathbf{N}$ *and for all* $f \in C^r[0, 1]$

$$\| e_{2p} (B_n^{(K)} f)^{(q+r)} \| \leqslant M n^{q - \min\{p, [q/2]\}} \| f^{(r)} \|;$$

(2) *for all* $\beta, \gamma \in \mathbf{N}_0$ $(\beta \leqslant \alpha)$ *and for all* $f \in C^{2\beta+\gamma}[0, 1]$,

$$\| (B_n^{(K)} f)^{(\gamma)} - f^{(\gamma)} \| = o(n^{-\beta}) \qquad (n \to \infty);$$

(3) *for all* $\gamma \in \mathbf{N}_0$ *and for all* $f \in C^{2\alpha+\gamma+2}[0,1]$,

$$\lim_{n\to\infty} n^{\alpha+1}((B_n^{(K)}f)^{(\gamma)} - f^{(\gamma)})$$

$$= \begin{cases} (-1)^\alpha (2\alpha+1)!! \, (\tfrac{2}{3}\alpha e_{2\alpha+1} f^{[2\alpha+1]} + e_{2\alpha+2} f^{[2\alpha+2]})^{(\gamma)} & \text{if} \quad K = 2\alpha, \\ (-1)^\alpha (2\alpha+1)!! \, (e_{2\alpha+2} f^{[2\alpha+2]})^{(\gamma)} & \text{if} \quad K = 2\alpha+1, \end{cases}$$

*in the sense of* $\|\cdot\|$.

*Proof.* Let $n \in \mathbf{N}$ and suppose $n > K$. We substitute $B_n^{(K)}$ into $T_n$ of Theorem 2.4 and identify the given $K$ with $K$ in the theorem. Then $K \geqslant 2\alpha$ and for every $k \in \mathbf{N}_0$,

$$V_k^n = \begin{cases} U_k^n & \text{if} \quad k \leqslant K, \\ 0 & \text{if} \quad K < k \leqslant n. \end{cases}$$

Thus the conditions (a) and (c) are trivial. We can also verify (b) using Theorem 2.3. Therefore, Theorem 2.4 implies the properties (1) and (2) in this corollary. The property (3) is also derived by calculating $R_k$ in Theorem 2.4 with the aid of Theorem 2.3. ∎

Now we compare this corollary with the preceding results. When $p = q = r = 0$, (1) reduces to

(1′) there exists a constant $M$ such that for all $n \in \mathbf{N}$ and for all $f \in C[0,1]$

$$\|B_n^{(K)} f\| \leqslant M \, \|f\|.$$

This is nothing but the result of [7]. Besides, when $\gamma = 0$, we can rewrite (3) as

(3′) for all $f \in C^{2\alpha+2}[0,1]$,

$$\lim_{n\to\infty} n^{\alpha+1}(B_n^{(K)} f - f)$$

$$= \begin{cases} \dfrac{(-1)^\alpha \, (4\alpha(\alpha+1) \, e_{2\alpha+1} f^{(2\alpha+1)} + 3 e_{2\alpha+2} f^{(2\alpha+2)})}{3 \cdot 2^{\alpha+1} (\alpha+1)!} & \text{if} \quad K = 2\alpha, \\ \dfrac{(-1)^\alpha \, e_{\alpha+2} f^{(2\alpha+2)}}{2^{\alpha+1} (\alpha+1)!} & \text{if} \quad K = 2\alpha+1, \end{cases}$$

in the sense of $\|\cdot\|$.

Here we used the identities $(2\alpha+1)! = (2\alpha+1)!! \, (2\alpha)!! = (2\alpha+1)!! \, 2^\alpha \alpha!$ and $(2\alpha+2)! = (2\alpha+2)!! \, (2\alpha+1)!! = 2^{\alpha+1}(\alpha+1)! \, (2\alpha+1)!!$. As the class $C^{2\alpha+3}[0,1]$ can be embedded into $C^{2\alpha+2}[0,1]$, by regarding the sense of convergence as pointwise, (3′) reduces further to the result of [6]. As we see from these facts, Corollary 2.1 itself is a much more general and detailed result than the preceding ones, and as to the theorems, therefore, all the more.

## 3. PROOFS OF THEOREMS 2.1–2.3

In this section, we prove the first three theorems in the previous section.

*Proof of Theorem* 2.1. Suppose the condition (2) holds. It is trivial that $Tf = \sum_{\nu=0}^{n} f(\nu/n)\, \tau_\nu$ $(\tau_\nu \in \mathbf{P}_n)$. Let $f \in \mathbf{P}_m$ $(0 \leqslant m \leqslant n)$. Then, as is well known, $B_n f \in \mathbf{P}_m$. Thus (2) ⇒ (1) is immediate.

Suppose the condition (1) holds. We fix $x \in [0, 1]$ for a while and expand with respect to $\xi \in [0, 1]$

$$B_n f(\xi) = \sum_{k=0}^{n} (\xi - x)^k \,(B_n f)^{[k]}\,(x).$$

Since it is well known that $B_n$ is invertible on $\mathbf{P}_n$ (e.g., [5–7]), we can calculate as

$$\begin{aligned} Tf(\xi) &= TL_n f(\xi) = TB_n^{-1} B_n L_n f(\xi) = TB_n^{-1} B_n f(\xi) \\ &= \sum_{k=0}^{n} TB_n^{-1}(\cdot - x)^k\,(\xi)(B_n f)^{[k]}\,(x). \end{aligned}$$

Letting $\xi = x$ gives

$$Tf(x) = \sum_{k=0}^{n} V_k(x)(B_n f)^{[k]}\,(x),$$

where $V_k(x) = TB_n^{-1}(\cdot - x)^k\,(x)$. Thus the existence of $V_k$ satisfying the above formula is guaranteed.

Let $x \in [0, 1]$, $t \in (-1, 1)$ and fix them for a while. We consider the case

$$f(\xi) = (1 + (1 - x)\, t)^{n\xi}\,(1 - xt)^{n(1-\xi)} \qquad (\xi \in [0, 1]).$$

Then

$$\begin{aligned} B_n f(\xi) &= \sum_{\nu=0}^{n} (1 + (1 - x))^\nu\,(1 - xt)^{n-\nu} \binom{n}{\nu} \xi^\nu (1 - \xi)^{n-\nu} \\ &= \sum_{\nu=0}^{n} \binom{n}{\nu} (\xi + (1 - x)\,\xi t)^\nu\,(1 - \xi - x(1 - \xi)\,t)^{n-\nu} \\ &= (1 + (\xi - x)\,t)^n. \end{aligned}$$

For all $k \leqslant n$,

$$(B_n f)^{[k]}(\xi) = \binom{n}{k} (1 + (\xi - x) t)^{n-k} t^k.$$

$$(B_n f)^{[k]}(x) = \binom{n}{k} t^k.$$

Therefore the relation $Tf = \sum_{k=0}^n V_k (B_n f)^{[k]}$ implies

$$\sum_{v=0}^n (1 + (1-x) t)^v (1 - xt)^{n-v} \tau_v(x) = \sum_{k=0}^n \binom{n}{k} V_k(x) t^k.$$

This means the $V_k$ are obtained by expanding the left-hand side with respect to $t$, and consequently the $V_k$ are unique. Generalizing the above formula, we expand for every $l \in \mathbf{N}_0$

$$\sum_{v=0}^n \left(\frac{v}{n} - x\right)^l (1 + (1-x) t)^v (1 - xt)^{n-v} \tau_v(x) = \sum_{k=0}^n \binom{n}{k} V_{k,l}(x) t^k.$$

Here we can identify $V_k$ with $V_{k,0}$. Differentiating by $t$ and multiplying by $(1 + (1-x) t)(1 - xt)$ both sides of the above equation, we get

$$(1 + (1-2x) t - Xt^2) \sum_{k=1}^n k \binom{n}{k} V_{k,l}(x) t^{k-1}$$
$$= n \sum_{k=0}^n \binom{n}{k} V_{k,l+1}(x) t^k - nXt \sum_{k=0}^n \binom{n}{k} V_{k,l}(x) t^k$$

by virtue of

$$(1 + (1-x) t)(1 - xt) \frac{d}{dt} [(1 + (1-x) t)^v (1 - xt)^{n-v}]$$
$$= ((v - nx) - nXt)(1 + (1-x) t)^v (1 - xt)^{n-v}.$$

Rearrangement of the above formula with the conventional definition $V_{-1,l}(x) = 0$ gives

$$\sum_{k=0}^{n-1} \binom{n}{k} (n-k) V_{k+1,l}(x) t^k$$
$$= \sum_{k=0}^n \binom{n}{k} (nV_{k,l+1}(x) - ke_1(x) V_{k,l}(x) - ke_2(x) V_{k-1,l}(x)) t^k.$$

Equating coefficients of $t^k$ on both sides yields

$$(n-k)\, V_{k+1,l} = nV_{k,l+1} - k(e_1 V_{k,l} + e_2 V_{k-1,l}) \qquad (0 \leqslant k \leqslant n-1,\ l \geqslant 0).$$

Since we need $V_{k,0}$ only, we may restrict the region where $l$ moves, to $0 \leqslant l \leqslant n-k-1$. In addition, the initial condition is

$$V_{-1,l} = 0 \qquad (0 \leqslant l \leqslant n-1)$$

and

$$V_{0,l}(x) = \sum_{\nu=0}^{n} \left(\frac{\nu}{n} - x\right)^l \tau_\nu(x) = T(\cdot - x)^l\,(x) \qquad (x \in [0,1]) \quad (0 \leqslant l \leqslant n),$$

derived by letting $t=0$ on both sides of the formula generating $V_{k,l}$.

Finally, we let $\varphi(\xi) = \xi$ ($\xi \in [0,1]$) and expand

$$V_{0,l}(x) = \sum_{m=0}^{l} \binom{l}{m} T\varphi^m(x) \cdot (-x)^{l-m}. \tag{3.1}$$

Then $\varphi^m \in \mathbf{P}_m$ and $T\mathbf{P}_m \subseteq \mathbf{P}_m$ $(0 \leqslant m \leqslant n)$ imply $V_{0,l} \in \mathbf{P}_l$ $(0 \leqslant l \leqslant n)$. Using the recursion formula, we obtain $V_k \in \mathbf{P}_k$ $(0 \leqslant k \leqslant n)$. Thus $(1) \Rightarrow (2)$ is proved. ∎

*Proof of Theorem* 2.2. Obviously, $T = L_n$ satisfies the condition (1) in Theorem 2.1, therefore it also satisfies (2). We define $U_k^n$ as $V_k$ in the case $T = L_n$. Then this theorem is immediate except the recursion formula.

When $T = L_n$, recalling (3.1), we can expand

$$V_{0,l}(x) = \sum_{m=0}^{l} \binom{l}{m} L_n\varphi^m(x) \cdot (-x)^{l-m} = \sum_{m=0}^{l} \binom{l}{m} x^m (-x)^{l-m}$$

$$= (x + (-x))^l = \begin{cases} 1 & (l=0), \\ 0 & (1 \leqslant l \leqslant n). \end{cases}$$

Thus, from the recursion formula in Theorem 2.1, the identities

$$V_{k,l+1} = 0 \qquad (0 \leqslant k \leqslant n-1,\ 0 \leqslant l \leqslant n-k-1)$$

hold. Then it suffices to consider the case $l=0$. ∎

*Proof of Theorem* 2.3. We prove this theorem by induction with the recursion formula in Theorem 2.2. It is valid when $k=0$ because $U_0^n = 1$ $(n \geqslant 0)$ and $U_1^n = 0$ $(n \geqslant 1)$. Assume this theorem is valid for a fixed $k \in \mathbf{N}_0$. Then for all $n \geqslant 2(k+1)$,

$$
\begin{aligned}
&(n-2k-1)\, U^n_{2(k+1)} \\
&\quad = -(2k+1)(e_1 U^n_{2k+1} + e_2 U^n_{2k}) \\
&\quad = -(2k+1)\left(e_1 \sum_{l=0}^{k} u_{2k+1,\,2l+1}(n)\, e_{2l+1} + e_2 \sum_{l=0}^{k} u_{2k,\,2l}(n)\, e_{2l}\right).
\end{aligned}
$$

Since $e_1 e_{2l+1} = e_{2l} - 4e_{2(l+1)}$ and $e_2 e_{2l} = e_{2(l+1)}$,

$$
\begin{aligned}
&(n-2k-1)\, U^n_{2(k+1)} \\
&\quad = -(2k+1)\left(\sum_{l=0}^{k} u_{2k+1,\,2l+1}(n)(e_{2l} - 4e_{2(l+1)}) + \sum_{l=0}^{k} u_{2k,\,2l}(n)\, e_{2(l+1)}\right) \\
&\quad = -(2k+1)\left(\sum_{l=0}^{k} u_{2k+1,\,2l+1}(n)\, e_{2l} - 4\sum_{l=1}^{k+1} u_{2k+1,\,2(l-1)+1}(n)\, e_{2l}\right. \\
&\qquad \left. + \sum_{l=1}^{k+1} u_{2k,\,2(l-1)}(n)\, e_{2l}\right).
\end{aligned}
$$

Here we compare the coefficients on both sides. It is obvious that

$$
u_{2(k+1),\,2l+1}(n) = 0 \qquad \text{if} \quad l \leqslant k.
$$

$$
\begin{aligned}
&(n-2k-1)\, u_{2(k+1),\,2l}(n) \\
&\quad = -(2k+1)\begin{cases}
u_{2k+1,\,1}(n) & \text{if} \quad l = 0, \\
(u_{2k+1,\,2l+1}(n) - 4u_{2k+1,\,2(l-1)+1}(n) + u_{2k,\,2(l-1)}(n)) & \\
\qquad & \text{if} \quad 1 \leqslant l \leqslant k, \\
(-4u_{2k+1,\,2k+1}(n) + u_{2k,\,2k}(n)) & \\
\qquad & \text{if} \quad l = k+1.
\end{cases}
\end{aligned}
$$

This recursion formula and the assumption of induction imply

$$
\begin{aligned}
u_{2(k+1),\,2l}(n) &= O(n^{-1})(O(n^{l-2k-1}) + O(n^{l-2k-2}) + O(n^{l-2k-1})) \\
&= O(n^{l-2(k+1)}) \qquad \text{if} \quad l \leqslant k.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\lim_{n\to\infty} n^{k+1} u_{2(k+1),\,2(k+1)}(n) \\
&\quad = \lim_{n\to\infty} n^k (n-2k-1)\, u_{2(k+1),\,2(k+1)}(n) \\
&\quad = \lim_{n\to\infty} (-(2k+1))(-4n^{-1} n^{k+1} u_{2k+1,\,2k+1}(n) + n^k u_{2k,\,2k}(n)) \\
&\quad = -(2k+1)(0 + (-1)^k\, (2k-1)!!) = (-1)^{k+1}\, (2(k+1)-1)!!.
\end{aligned}
$$

Thus the estimation of $u_{2(k+1),l}(n)$ $(0 \leqslant l \leqslant 2(k+1))$ is demonstrated and that of $u_{2(k+1)+1,l}(n)$ $(0 \leqslant l \leqslant 2(k+1)+1)$ is similarly shown by using the assumption of induction and the consequence on $u_{2(k+1),l}(n)$. ∎

## 4. PRELIMINARY LEMMAS FOR THE PROOF OF THEOREM 2.4

This section is devoted to preparation of lemmas indispensable to prove Theorem 2.4.

LEMMA 4.1. *Let $\{V_k^n\}_{n=k}^{\infty}$ be a sequence of polynomials of degree at most $k \in \mathbf{N}_0$. For each $n \in \mathbf{N}$, we expand*

$$V_k^n = \sum_{l=0}^{k} v_{k,l}(n)\, e_l,$$

*and furthermore for each $r \in \mathbf{N}_0$ $(r \leqslant k)$*

$$(V_k^n)^{(r)} = \sum_{l=0}^{k-r} v_{k,l}^r(n)\, e_l.$$

*We suppose*

$$v_{k,l}(n) = O(n^{[l/2]-k}) \qquad (n \to \infty) \quad (0 \leqslant l \leqslant k).$$

*Then*

$$v_{k,l}^r(n) = O(n^{\min\{[l/2]+r,\,[k/2]\}-k}) \qquad (n \to \infty) \quad (0 \leqslant l \leqslant k-r).$$

*Proof.* We prove this lemma by induction. It is trivial when $r=0$. Assume this lemma is valid for a fixed $r \in \mathbf{N}_0$ $(r \leqslant k-1)$. Then

$$\begin{aligned}(V_k^n)^{(r+1)} &= \sum_{l=0}^{k-r} v_{k,l}^r(n)\, e_l' \\ &= \sum_{l=0}^{[(k-r)/2]} v_{k,2l}^r(n)\, e_{2l}' + \sum_{l=0}^{[(k-r-1)/2]} v_{k,2l+1}^r(n)\, e_{2l+1}'.\end{aligned}$$

Since

$$e_0' = 0, \qquad e_{2l}' = l e_{2l-1} \qquad (l \geqslant 1)$$

and

$$e_1' = -2e_0', \qquad e_{2l+1}' = l e_{2(l-1)} - 2(2l+1)\, e_{2l} \qquad (l \geqslant 1),$$

we have

$$(V_k^n)^{(r+1)} = \sum_{l=1}^{[(k-r)/2]} l v_{k,2l}^r(n)\, e_{2l-1} - 2v_{k,1}^r(n)\, e_0 + \sum_{l=1}^{[(k-r-1)/2]} v_{k,2l+1}^r(n)(l e_{2(l-1)} - 2(2l+1)\, e_{2l}).$$

Therefore,

$$\sum_{l=0}^{[(k-r-1)/2]} v_{k,2l}^{r+1}(n)\, e_{2l} + \sum_{l=0}^{[(k-r)/2]-1} v_{k,2l+1}^{r+1}(n)\, e_{2l+1}$$
$$= \sum_{l=0}^{[(k-r-1)/2]-1} (l+1)\, v_{k,2l+3}^r(n)\, e_{2l} - 2 \sum_{l=0}^{[(k-r-1)/2]} (2l+1)\, v_{k,2l+1}^r(n)\, e_{2l} + \sum_{l=0}^{[(k-r)/2]-1} (l+1)\, v_{k,2l+2}^r(n)\, e_{2l+1}.$$

Equating coefficients of $e_{2l}$ and $e_{2l+1}$ on both sides yields

$$v_{k,2l}^{r+1}(n) = \begin{cases} (l+1)\, v_{k,2l+3}^r(n) - 2(2l+1)\, v_{k,2l+1}^r(n) \\ \qquad (0 \leqslant l \leqslant [(k-r-1)/2]-1) \\ -2(2l+1)\, v_{k,2l+1}^r(n) \\ \qquad (l = [(k-r-1)/2]) \end{cases}$$

and

$$v_{k,2l+1}^{r+1}(n) = (l+1)\, v_{k,2l+2}^r(n) \qquad (0 \leqslant l \leqslant [(k-r)/2]-1).$$

These recursion formulas and the assumption of induction imply

$$v_{k,l}^{r+1}(n) = O(n^{\min\{[l/2]+(r+1),\,[k/2]\}-k}) \quad (n \to \infty) \ \ (0 \leqslant l \leqslant k-(r+1)). \quad \blacksquare$$

LEMMA 4.2. *For all* $n \in \mathbf{N}$, $k \in \mathbf{N}_0$ $(k \leqslant n)$ *and for all* $f\colon [0,1] \to \mathbf{R}$,

$$(B_n f)^{(k)} = n^{(k)} \sum_{v=0}^{n-k} \Delta_{1/n}^k f\left(\frac{v}{n}\right) b_{n-k,v},$$

*where*

$$b_{n,v}(x) = \binom{n}{v} x^v (1-x)^{n-v} \qquad (x \in [0,1]).$$

*Remark.* We use this notation $b_{n,v}$ throughout the paper.

A proof of this lemma appears in [1, p. 12].

LEMMA 4.3. *For all $p, q, r \in \mathbf{N}_0$, there exists a constant $M$ such that for all $n \in \mathbf{N}$ and for all $f \in C^r[0, 1]$*

$$\|e_p (B_n f)^{(q+r)}\| \leqslant M n^{q - \min\{[p/2], [q/2]\}} \|f^{(r)}\|.$$

*Proof.* It was shown in [3, Lemma 3.5; 4, Theorem 9.4.1] that for all $p \in \mathbf{N}_0$, there exists a constant $M$ such that for all $n \in \mathbf{N}$ and for all $f \in C[0, 1]$

$$\|e_{2p} (B_n f)^{(2p)}\| \leqslant M n^p \|f\|,$$

that is,

$$\|e_{2p} D^{2p} B_n\| \leqslant M n^p. \tag{4.1}$$

By considering the Lebesgue constant of the operator $e_{2p} D^q B_n$, we get

$$\|e_{2p} D^q B_n\| = \left\| e_{2p} \sum_{\nu=0}^{n} |b_{n,\nu}{}^{(q)}| \right\| \qquad \text{for all} \quad p, q \in \mathbf{N}_0 \text{ and } n \in \mathbf{N}. \tag{4.2}$$

We can assume $n > s \in \mathbf{N}_0$ without loss of generality. Applying Lemma 4.2, (4.2), and (4.1), we can estimate

$$\begin{aligned} \|e_{2p} D^{2p+s} B_n\| &= \max_{\|g\|=1} \|e_{2p}((B_n g)^{(s)})^{(2p)}\| \\ &= \max_{\|g\|=1} \left\| e_{2p} \cdot n^{(s)} \sum_{\nu=0}^{n-s} \Delta_{1/n}^{s} g\left(\frac{\nu}{n}\right) b_{n-s,\nu}{}^{(2p)} \right\| \\ &\leqslant 2^s n^{(s)} \left\| e_{2p} \sum_{\nu=0}^{n-s} |b_{n-s,\nu}{}^{(2p)}| \right\| \\ &= 2^s n^{(s)} \|e_{2p} D^{2p} B_{n-s}\| \leqslant M' n^{s+p}. \end{aligned} \tag{4.3}$$

Let $f \in C^r[0, 1]$. We can assume $n > r$ without loss of generality. Applying Lemma 4.2, the mean value theorem, and (4.2), we can calculate as

$$(B_n f)^{(q+r)} = ((B_n f)^{(r)})^{(q)} = n^{(r)} \sum_{\nu=0}^{n-r} \Delta_{1/n}^{r} f\left(\frac{\nu}{n}\right) b_{n-r,\nu}{}^{(q)},$$

$$\|e_p (B_n f)^{(q+r)}\| \leqslant \|f^{(r)}\| \left\| e_p \sum_{\nu=0}^{n-r} |b_{n-r,\nu}{}^{(q)}| \right\| = \|f^{(r)}\| \cdot \|e_p D^q B_{n-r}\|.$$

Replacing $p$ by $\min\{[p/2], [q/2]\}$ and letting $s = q - 2\min\{[p/2], [q/2]\}$ in (4.3) imply

$$\begin{aligned}\|e_p D^q B_{n-r}\| &\leqslant \|e_{p-2\min\{[p/2],[q/2]\}}\| \cdot \|e_{2\min\{[p/2],[q/2]\}} D^q B_{n-r}\| \\ &\leqslant M n^{q-\min\{[p/2],[q/2]\}},\end{aligned}$$

where $M$ is a suitable constant. ∎

LEMMA 4.4. *Let* $r, s \in \mathbf{N}_0$, $f \in C^{r+s}[0,1]$, *and for each* $x \in [0,1]$

$$g_x(\xi) = \sum_{j=0}^{r+s} f^{[j]}(x)(\xi - x)^j \qquad (\xi \in [0,1]), \quad h_x = f - g_x.$$

*Then*

$$\max_{x \in [0,1]} |(B_n h_x)^{(r)}(x)| = o(n^{-s/2}) \qquad (n \to \infty).$$

*Proof.* Let $n \in \mathbf{N}$. We can assume $n > r$ without loss of generality. Lemma 4.2 and the mean value theorem imply

$$\begin{aligned}(B_n h_x)^{(r)}(x) &= n^{(r)} \sum_{v=0}^{n-r} \Delta^r_{1/n} h_x\left(\frac{v}{n}\right) b_{n-r,v}(x) \\ &= \frac{n^{(r)}}{n^r} \sum_{v=0}^{n-r} {h_x}^{(r)}\left(\frac{v + r\theta_v}{n}\right) b_{n-r,v}(x) \quad (0 < \theta_0, \theta_1, ..., \theta_{n-r} < 1).\end{aligned}$$

Applying Taylor's theorem to $f^{(r)}$, we obtain

$$\begin{aligned}(h_x)^{(r)}(\xi) &= f^{(r)}(\xi) - {g_x}^{(r)}(\xi) \\ &= f^{(r)}(\xi) - \sum_{j=0}^{s} (f^{(r)})^{[j]}(x)(\xi - x)^j \\ &= \frac{f^{(r+s)}(x + \lambda(\xi - x)) - f^{(r+s)}(x)}{s!} (\xi - x)^s \quad \text{for some } \lambda \in (0, 1],\end{aligned}$$

where we noticed that $s = 0$ yields $\lambda = 1$. Since $f^{(r+s)}$ is continuous on $[0,1]$, it is uniformly continuous on $[0,1]$. Take an arbitrary $\varepsilon > 0$. We can find a $\delta > 0$ such that for all $x_1, x_2 \in [0,1]$

$$|x_2 - x_1| < \delta \qquad \text{implies} \quad |f^{(r+s)}(x_2) - f^{(r+s)}(x_1)| < \varepsilon.$$

Therefore, when $|(v + r\theta_v)/n - x| < \delta$,

$$\left| {h_x}^{(r)}\left(\frac{v + r\theta_v}{n}\right)\right| \leqslant \frac{\varepsilon}{s!} \left|\frac{v + r\theta_v}{n} - x\right|^s.$$

When $|(v+r\theta_v)/n - x| \geqslant \delta$,

$$\left| h_x{}^{(r)}\left(\frac{v+r\theta_v}{n}\right)\right| \leqslant \frac{H}{s!\,\delta}\left|\frac{v+r\theta_v}{n}-x\right|^{s+1},$$

where $H = 2\,\|f^{(r+s)}\|$. Hence in either case,

$$\left| h_x{}^{(r)}\left(\frac{v+r\theta_v}{n}\right)\right| \leqslant \frac{\varepsilon}{s!}\left|\frac{v+r\theta_v}{n}-x\right|^{s} + \frac{H}{s!\,\delta}\left|\frac{v+r\theta_v}{n}-x\right|^{s+1}.$$

Now we can calculate as

$$\begin{aligned} |(B_n h_x)^{(r)}(x)| &\leqslant \sum_{v=0}^{n-r} \left| h_x{}^{(r)}\left(\frac{v+r\theta_v}{n}\right)\right| b_{n-r,\,v}(x) \\ &\leqslant \frac{\varepsilon}{s!}\sum_{v=0}^{n-r}\left|\frac{v+r\theta_v}{n}-x\right|^{s} b_{n-r,\,v}(x) \\ &\quad + \frac{H}{s!\,\delta}\sum_{v=0}^{n-r}\left|\frac{v+r\theta_v}{n}-x\right|^{s+1} b_{n-r,\,v}(x). \end{aligned}$$

Since $0 < \theta_v < 1$ and $0 \leqslant v/(n-r) \leqslant 1$ imply $|\theta_v - v/(n-r)| < 1$,

$$\left|\frac{v+r\theta_v}{n}-x\right| = \left|\left(\frac{v}{n-r}-x\right)+\frac{r}{n}\left(\theta_v-\frac{v}{n-r}\right)\right| \leqslant \left|\frac{v}{n-r}-x\right| + \frac{r}{n}.$$

It was shown in [1, pp. 13–15] that

$$\max_{x\in[0,\,1]}\sum_{v=0}^{n}\left|\frac{v}{n}-x\right|^{s} b_{n,\,v}(x) = O(n^{-s/2}) \qquad (n\to\infty).$$

Using this fact, we can estimate

$$\begin{aligned} &\max_{x\in[0,\,1]}\sum_{v=0}^{n-r}\left|\frac{v+r\theta_v}{n}-x\right|^{s} b_{n-r,\,v}(x) \\ &\leqslant \max_{x\in[0,\,1]}\sum_{v=0}^{n-r}\left(\left|\frac{v}{n-r}-x\right|+\frac{r}{n}\right)^{s} b_{n-r,\,v}(x) \\ &= \max_{x\in[0,\,1]}\sum_{v=0}^{n-r} b_{n-r,\,v}(x)\sum_{m=0}^{s}\binom{s}{m}\left|\frac{v}{n-r}-x\right|^{m}\left(\frac{r}{n}\right)^{s-m} \\ &\leqslant \sum_{m=0}^{s}\binom{s}{m}\left(\frac{r}{n}\right)^{s-m}\max_{x\in[0,\,1]}\sum_{v=0}^{n-r}\left|\frac{v}{n-r}-x\right|^{m} b_{n-r,\,v}(x) \\ &= \sum_{m=0}^{s} O(n^{-s+m})\,O(n^{-m/2}) = \sum_{m=0}^{s} O(n^{-s+m/2}) = O(n^{-s/2}). \end{aligned}$$

Therefore,

$$\max_{x\in[0,1]} |(B_n h_x)^{(r)}(x)| \leqslant \frac{\varepsilon}{s!} M_1 n^{-s/2} + \frac{H}{s!\,\delta} M_2 n^{-(s+1)/2} \quad \text{for some} \quad M_1, M_2 > 0$$

$$< Mn^{-s/2}\varepsilon \qquad \text{for all sufficiently large } n,$$

where $M$ is a suitable constant. ∎

Note that some special cases of Lemmas 4.3 and 4.4 are in Theorems 9.4.1 and 9.7.1 and in Lemma 9.5.2 in [4].

## 5. PROOF OF THEOREM 2.4

Now we are to prove Theorem 2.4. Here the notations Theorem 2.4(1), (2), and (3) stand for the properties (1), (2), and (3), respectively, in Theorem 2.4.

*Proof of Theorem* 2.4(1). We can assume $n > K$ without loss of generality. From the relation $T_n f = \sum_{k=0}^{K} V_k^n (B_n f)^{[k]}$, we expand

$$\begin{aligned} e_{2p}(T_n f)^{(q+r)} &= e_{2p} \sum_{k=0}^{K} \sum_{m=0}^{q+r} \binom{q+r}{m} (V_k^n)^{(m)} ((B_n f)^{[k]})^{(q+r-m)} \\ &= \sum_{m=0}^{q+r} \binom{q+r}{m} \sum_{k=m}^{K} \frac{1}{k!} \left( \sum_{l=0}^{k-m} v_{k,l}^m(n)\, e_l \right) e_{2p}(B_n f)^{(k+q+r-m)} \\ &= \sum_{m=0}^{q+r} \binom{q+r}{m} \sum_{k=m}^{K} \frac{1}{k!} \sum_{l=0}^{k-m} v_{k,l}^m(n)\, e_{2p+l}(B_n f)^{(q+k-m+r)}. \end{aligned}$$

Applying Lemma 4.1, we have

$$v_{k,l}^m(n) = O(n^{[l/2]+m-k}).$$

Replacing $p$ by $2p+l$ and $q$ by $q+k-m$ in Lemma 4.3 implies

$$\|e_{2p+l}(B_n f)^{(q+k-m+r)}\| \leqslant Mn^{q+k-m-\min\{p+[l/2],\,[(q+k-m)/2]\}} \|f^{(r)}\|.$$

Thus

$$\|e_{2p}(T_n f)^{(q+r)}\| = \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^{K} \sum_{l=0}^{k-m} O(n^{[l/2]+m-k}) \times O(n^{q+k-m-\min\{p+[l/2],\,[(q+k-m)/2]\}})$$

$$= \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^{K} \sum_{l=0}^{k-m} O(n^{q-\min\{p,\,[(q+k-m)/2]-[l/2]\}})$$

$$= \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^{K} O(n^{q-\min\{p,\,[(q+k-m)/2]-[(k-m)/2]\}})$$

$$= \|f^{(r)}\| \sum_{m=0}^{q+r} \sum_{k=m}^{K} O(n^{q-\min\{p,\,[q/2]\}})$$

$$= \|f^{(r)}\|\ O(n^{q-\min\{p,\,[q/2]\}}) \qquad (n \to \infty),$$

that is,

$$\|e_{2p}(T_n f)^{(q+r)}\| \leqslant M n^{q-\min\{p,\,[q/2]\}}\ \|f^{(r)}\|,$$

where $M$ is a suitable constant and we used the inequality $[q/2]+[(k-m)/2] \leqslant [(q+k-m)/2]$ in the above calculation. ∎

*Proof of Theorem* 2.4(2). First, we give the proof in the case $f \in C^{K+\gamma}[0,1]$. We define the functions $g_x$, $h_x$ dependent of $x \in [0,1]$ as

$$g_x(\xi) = \sum_{j=0}^{K+\gamma} f^{[j]}(x)(\xi - x)^j \ \ (\xi \in [0,1]), \qquad h_x = f - g_x.$$

We can assume $n > K+\gamma$ without loss of generality. Since $\deg g_x \leqslant K+\gamma$,

$$(L_n g_x)^{(\gamma)}(\xi) = g_x{}^{(\gamma)}(\xi) = \sum_{j=\gamma}^{K+\gamma} f^{[j]}(x)\ j^{(\gamma)}(\xi - x)^{j-\gamma}.$$

Thus

$$(L_n g_x)^{(\gamma)}\,(x) = f^{[\gamma]}(x)\ \gamma! = f^{(\gamma)}(x).$$

Using this relation, we can estimate

$$\begin{aligned}
&\|(T_n f)^{(\gamma)} - f^{(\gamma)}\| \\
&\quad = \max_{x \in [0,1]} |(T_n f)^{(\gamma)}\,(x) - f^{(\gamma)}(x)| \\
&\quad = \max_{x \in [0,1]} |(T_n g_x)^{(\gamma)}\,(x) + (T_n h_x)^{(\gamma)}\,(x) - f^{(\gamma)}(x)| \\
&\quad \leqslant \max_{x \in [0,1]} |(T_n g_x)^{(\gamma)}\,(x) - f^{(\gamma)}(x)| + \max_{x \in [0,1]} |(T_n h_x)^{(\gamma)}\,(x)| \\
&\quad = \max_{x \in [0,1]} |(T_n g_x)^{(\gamma)}\,(x) - (L_n g_x)^{(\gamma)}\,(x)| + \max_{x \in [0,1]} |(T_n h_x)^{(\gamma)}\,(x)| \\
&\quad \leqslant \max_{x \in [0,1]} \|(T_n g_x)^{(\gamma)} - (L_n g_x)^{(\gamma)}\| + \max_{x \in [0,1]} |(T_n h_x)^{(\gamma)}\,(x)|.
\end{aligned}$$

Here

$$T_n g_x = \sum_{k=0}^{K} V_k^n (B_n g_x)^{[k]}$$

implies

$$(T_n g_x)^{(\gamma)} = \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^{K} \frac{1}{k!} (V_k^n)^{(m)} (B_n g_x)^{(k+\gamma-m)}.$$

Since deg $g_x \leqslant K+\gamma$ implies deg $B_n g_x \leqslant K+\gamma$,

$$L_n g_x = \sum_{k=0}^{K+\gamma} U_k^n (B_n g_x)^{[k]},$$

and consequently,

$$(L_n g_x)^{(\gamma)} = \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^{K+\gamma} \frac{1}{k!} (U_k^n)^{(m)} (B_n g_x)^{(k+\gamma-m)}.$$

Therefore,

$$\begin{aligned}
&\max_{x\in[0,1]} \|(T_n g_x)^{(\gamma)} - (L_n g_x)^{(\gamma)}\| \\
&\quad \leqslant \sum_{m=0}^{\gamma} \binom{\gamma}{m} \Bigg( \sum_{k=m}^{K} \frac{1}{k} \|(V_k^n)^{(m)} - (U_k^n)^{(m)}\| \max_{x\in[0,1]} \|(B_n g_x)^{(k+\gamma-m)}\| \\
&\qquad + \sum_{k=K+1}^{K+\gamma} \frac{1}{k!} \|(U_k^n)^{(m)}\| \max_{x\in[0,1]} \|(B_n g_x)^{(k+\gamma-m)}\| \Bigg).
\end{aligned}$$

It follows from the condition (c) and Markov's inequality that

$$\|(V_k^n)^{(m)} - (U_k^n)^{(m)}\| = o(n^{-\alpha}).$$

It follows from Theorem 2.3 and Markov's inequality that

$$\|(U_k^n)^{(m)}\| = O(n^{[k/2]-k}).$$

Furthermore, applying Lemma 4.3 with letting $p=q=0$ and $r=k+\gamma-m$, we get

$$\|(B_n g_x)^{(k+\gamma-m)}\| \leqslant M \; \|g_x{}^{(k+\gamma-m)}\| \qquad \text{for some constant } M.$$

Since

$$g_x{}^{(k+\gamma-m)}(\xi) = \sum_{j=k+\gamma-m}^{K+\gamma} f^{[j]}(x)\ j^{(k+\gamma-m)}(\xi-x)^{j-k-\gamma+m},$$

$$\|g_x{}^{(k+\gamma-m)}\| \leqslant \sum_{j=k+\gamma-m}^{K+\gamma} j^{(k+\gamma-m)}\ |f^{[j]}(x)|$$

$$\leqslant \sum_{j=k+\gamma-m}^{K+\gamma} j^{(k+\gamma-m)}\ \|f^{[j]}\|.$$

Thus

$$\max_{x\in[0,1]} \|(B_n g_x)^{(k+\gamma-m)}\| \leqslant M' \qquad \text{for some constant } M'.$$

Consequently,

$$\max_{x\in[0,1]} \|(T_n g_x)^{(\gamma)} - (L_n g_x)^{(\gamma)}\| = o(n^{-\alpha}) + \sum_{k=K+1}^{K+\gamma} O(n^{[k/2]-k})$$

$$= o(n^{-\alpha}) + O(n^{[(K+1)/2]-(K+1)}) = o(n^{-\alpha}),$$

where we used the assumption $K \geqslant 2\alpha$. On the other hand,

$$T_n h_x = \sum_{k=0}^{K} V_k^n (B_n h_x)^{[k]}$$

implies

$$(T_n h_x)^{(\gamma)} = \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^{K} \frac{1}{k!} (V_k^n)^{(m)} (B_n h_x)^{(k+\gamma-m)}.$$

Therefore,

$$\max_{x\in[0,1]} |(T_n h_x)^{(\gamma)}(x)|$$

$$\leqslant \sum_{m=0}^{\gamma} \binom{\gamma}{m} \sum_{k=m}^{K} \frac{1}{k!} \|(V_k^n)^{(m)}\| \max_{x\in[0,1]} |(B_n h_x)^{(k+\gamma-m)}(x)|.$$

It follows from the condition (b)—accordingly $\|V_k^n\| = O(n^{[k/2]-k})$—and Markov's inequality that

$$\|(V_k^n)^{(m)}\| = O(n^{[k/2]-k}).$$

Furthermore, applying Lemma 4.4 with $r=k+\gamma-m$ and $s=K-k+m$, we get

$$\max_{x\in[0,1]} |(B_n h_x)^{(k+\gamma-m)}(x)| = o(n^{-(K-k+m)/2}).$$

Consequently,

$$\max_{x\in[0,1]} |(T_n h_x)^{(\gamma)}(x)| = \sum_{m=0}^{\gamma} \sum_{k=m}^{K} O(n^{[k/2]-k})\, o(n^{-(K-k+m)/2})$$

$$= \sum_{m=0}^{\gamma} o(n^{-(K+m)/2}) = o(n^{-K/2}) = o(n^{-\alpha}).$$

Hence we obtain

$$\|(T_n f)^{(\gamma)} - f^{(\gamma)}\| = o(n^{-\alpha}) \qquad (n\to\infty) \quad \text{for all } f\in C^{K+\gamma}[0,1]. \tag{5.1}$$

Next, we give the proof in the case $f\in C^{2\beta+\gamma}[0,1]$. It is well known (see [1, pp. 25–26]) that for all $r\in\mathbf{N}_0$ and for all $f\in C^r[0,1]$

$$\lim_{n\to\infty} \|(B_n f)^{(r)} - f^{(r)}\| = 0.$$

(We can also prove it by applying (5.1) with $T_n = B_n$, $\alpha=0$, $K=0$, $\gamma=r$.) Take an arbitrary $\varepsilon>0$. Then there exists an $N\in\mathbf{N}$ such that

$$\|(B_N f)^{(r)} - f^{(r)}\| < \varepsilon \qquad (r\leqslant 2\beta+\gamma).$$

Let $\varphi = B_N f$ and $\rho = f-\varphi$. Then

$$\|\rho^{(r)}\| < \varepsilon \qquad (r\leqslant 2\beta+\gamma).$$

We define the new operator ${}_\beta T_n$ as

$${}_\beta T_n f = \sum_{k=0}^{2\beta} V_k^n (B_n f)^{[k]} \qquad (f\colon [0,1]\to\mathbf{R}).$$

Since

$$T_n f - f = T_n\rho - {}_\beta T_n \rho + {}_\beta T_n \rho - \rho + T_n\varphi - \varphi,$$

we can estimate

$$\|(T_n f)^{(\gamma)} - f^{(\gamma)}\| \leqslant \|(T_n\rho)^{(\gamma)} - ({}_\beta T_n\rho)^{(\gamma)}\|$$
$$+ \|({}_\beta T_n\rho)^{(\gamma)} - \rho^{(\gamma)}\| + \|(T_n\varphi)^{(\gamma)} - \varphi^{(\gamma)}\|.$$

Since $\varphi$ is a polynomial, it is immediate from (5.1) that

$$\|(T_n\varphi)^{(\gamma)} - \varphi^{(\gamma)}\| = o(n^{-\alpha}).$$

Applying (5.1) and replacing $T_n$ by ${}_\beta T_n$, $\alpha$ by $\beta$, and $K$ by $2\beta$, we have

$$\|({}_\beta T_n\rho)^{(\gamma)} - \rho^{(\gamma)}\| = o(n^{-\beta}).$$

Therefore, it suffices to estimate the first term of the right-hand side in the above inequality. Since

$$T_n\rho - {}_\beta T_n\rho = \sum_{k=2\beta+1}^{K} V_k^n(B_n\rho)^{[k]},$$

$$\begin{aligned}(T_n\rho)^{(\gamma)} - ({}_\beta T_n\rho)^{(\gamma)} &= \sum_{m=0}^{\gamma}\binom{\gamma}{m}\sum_{k=\{m,\,2\beta+1\}}^{K}\frac{1}{k!}(V_k^n)^{(m)}(B_n\rho)^{(k+\gamma-m)}\\ &= \sum_{m=0}^{\gamma}\binom{\gamma}{m}\left(\sum_{k=\{m,\,2\beta+1\}}^{2\beta+m}\frac{1}{k!}(V_k^n)^{(m)}(B_n\rho)^{(k+\gamma-m)}\right.\\ &\quad\left.+\sum_{k=2\beta+m+1}^{K}\frac{1}{k!}(V_k^n)^{(m)}(B_n\rho)^{(k+\gamma-m)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|(T_n\rho)^{(\gamma)} - ({}_\beta T_n\rho)^{(\gamma)}\| &\leqslant \sum_{m=0}^{\gamma}\binom{\gamma}{m}\sum_{k=\{m,\,2\beta+1\}}^{2\beta+m}\frac{1}{k!}\|(V_k^n)^{(m)}\|\,\|(B_n\rho)^{(k+\gamma-m)}\|\\ &\quad+\sum_{m=0}^{\gamma}\binom{\gamma}{m}\sum_{k=2\beta+m+1}^{K}\frac{1}{k!}\|(V_k^n)^{(m)}(B_n\rho)^{(k+\gamma-m)}\|.\end{aligned}$$

Applying Lemma 4.3 and letting $p=q=0$ and $r=k+\gamma-m$, we get

$$\begin{aligned}&\sum_{m=0}^{\gamma}\binom{\gamma}{m}\sum_{k=\{m,\,2\beta+1\}}^{2\beta+m}\frac{1}{k!}\|(V_k^n)^{(m)}\|\,\|(B_n\rho)^{(k+\gamma-m)}\|\\ &\quad=\sum_{m=0}^{\gamma}\sum_{k=\{m,\,2\beta+1\}}^{2\beta+m}O(n^{[k/2]-k})\,\|\rho^{(k+\gamma-m)}\| = O(n^{-\beta-1})\,\varepsilon.\end{aligned}$$

On the other hand, when $k\geqslant 2\beta+m+1$,

$$(V_k^n)^{(m)}(B_n\rho)^{(k+\gamma-m)} = \sum_{l=0}^{k-m} v_{k,l}^m(n)\,e_l(B_n\rho)^{(k+\gamma-m)}.$$

As we mentioned in the proof of Theorem 2.4(1), we have

$$v_{k,l}^m(n) = O(n^{[l/2]+m-k}).$$

Applying Lemma 4.3 and letting $p=l$, $q=k-2\beta-m$, and $r=2\beta+\gamma$, we get

$$\|e_l(B_n\rho)^{(k+\gamma-m)}\| = O(n^{k-2\beta-m-\min\{[l/2],\,[(k-m)/2]-\beta\}})\ \|\rho^{(2\beta+\gamma)}\|.$$

Therefore,

$$\begin{aligned}
&\sum_{m=0}^{\gamma}\binom{\gamma}{m}\sum_{k=2\beta+m+1}^{K}\frac{1}{k!}\,\|(V_k^n)^{(m)}\,(B_n\rho)^{(k+\gamma-m)}\| \\
&\quad=\sum_{m=0}^{\gamma}\ \sum_{k=2\beta+m+1}^{K}\ \sum_{l=0}^{k-m} O(n^{[l/2]+m-k}) \\
&\qquad\times O(n^{k-2\beta-m-\min\{[l/2],\,[(k-m)/2]-\beta\}})\ \|\rho^{(2\beta+\gamma)}\| \\
&\quad=\sum_{m=0}^{\gamma}\ \sum_{k=2\beta+m+1}^{K}\ \sum_{l=0}^{k-m} O(n^{[l/2]-2\beta-\min\{[l/2],\,[(k-m)/2]-\beta\}})\,\varepsilon \\
&\quad=\sum_{m=0}^{\gamma}\ \sum_{k=2\beta+m+1}^{K} O(n^{[(k-m)/2]-2\beta-\min\{[(k-m)/2],\,[(k-m)/2]-\beta\}})\,\varepsilon \\
&\quad=O(n^{-\beta})\,\varepsilon.
\end{aligned}$$

Thus the proof is completed. ∎

*Proof of Theorem* 2.4(3). We define the new operator $\tilde{T}_n$ as

$$\tilde{T}_n f = \sum_{k=0}^{n} \tilde{V}_k^n(B_n f)^{[k]} \qquad (f\colon [0,1]\to\mathbf{R}),$$

where

$$\tilde{V}_k^n=\begin{cases} V_k^n - n^{-(\alpha+1)}R_k & \text{if}\quad 0\leqslant k\leqslant 2\alpha+2,\\ V_k^n & \text{if}\quad 2\alpha+2<k\leqslant n.\end{cases}$$

Let $\tilde{K}=\max\{K, 2\alpha+2\}$. In Theorem 2.4(2), we replace $T_n$ by $\tilde{T}_n$, $\alpha$ by $\alpha+1$, and $K$ by $\tilde{K}$. Then we can easily verify that all the preconditions are satisfied. Therefore, we obtain for all $f\in C^{2\alpha+2+\gamma}$,

$$\|(\tilde{T}_n f)^{(\gamma)}-f^{(\gamma)}\| = o(n^{-(\alpha+1)}).$$

Now we can estimate

$$\left\| n^{\alpha+1}((T_n f)^{(\gamma)} - f^{(\gamma)}) - \left( \sum_{k=0}^{2\alpha+2} R_k f^{[k]} \right)^{(\gamma)} \right\|$$

$$= \left\| n^{\alpha+1}((\tilde{T}_n f)^{(\gamma)} - f^{(\gamma)}) + n^{\alpha+1}((T_n f)^{(\gamma)} - (\tilde{T}_n f)^{(\gamma)}) - \left( \sum_{k=0}^{2\alpha+2} R_k f^{[k]} \right)^{(\gamma)} \right\|$$

$$\leqslant n^{\alpha+1} \|(\tilde{T}_n f)^{(\gamma)} - f^{(\gamma)}\| + \sum_{k=0}^{2\alpha+2} \|(R_k (B_n f)^{[k]})^{(\gamma)} - (R_k f^{[k]})^{(\gamma)}\|.$$

As we mentioned above, the first term converges to zero when $n$ tends to infinity. It suffices to estimate the second term. It is equal to

$$\sum_{k=0}^{2\alpha+2} \|(R_k((B_n f)^{[k]} - f^{[k]}))^{(\gamma)}\|$$

$$= \sum_{k=0}^{2\alpha+2} \left\| \sum_{m=0}^{\gamma} \binom{\gamma}{m} \frac{R_k{}^{(\gamma-m)}}{k!} ((B_n f)^{(k+m)} - f^{(k+m)}) \right\|$$

$$\leqslant \sum_{k=0}^{2\alpha+2} \sum_{m=0}^{\gamma} \binom{\gamma}{m} \frac{\|R_k{}^{(\gamma-m)}\|}{k!} \|(B_n f)^{(k+m)} - f^{(k+m)}\| \to 0 \qquad (n \to \infty),$$

where we used (5.2). ∎

In this way, we have proved all the results.

In forthcoming papers, by using the theoretical results developed above, we will describe new specific classes of operators, which differ from those of Sablonnière, and are more convenient for practical applications.

## ACKNOWLEDGMENT

I thank very much Professor Paul Sablonnière, who gave me many helpful comments on the earlier draft.

## REFERENCES

1. G. G. Lorentz, "Bernstein Polynomials," 2nd ed., Chelsea, New York, 1986.
2. H. H. Gonska and J. Meier, A bibliography on approximation of functions by Bernstein type operators, *in* "Approximation Theory, IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 739–785, Academic Press, San Diego, 1983.
3. Z. Ditzian, A global inverse theorem for combinations of Bernstein polynomials, *J. Approx. Theory* **26** (1979), 277–292.
4. Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer Series in Computational Mathematics, Vol. 9, Springer-Verlag, Berlin/Heidelberg/New York, 1987.

5. P. Sablonnière, Bernstein quasi-interpolants on [0, 1], *in* "Multivariate Approximation Theory, Vol. IV" (C. K. Chui, W. Schempp, and K. Zeller, Eds.), pp. 287–294, Birkhäuser, Basel, 1989.
6. P. Sablonnière, A family of Bernstein quasi-interpolants on [0, 1], *Approx. Theory Appl.* **8**, No. 3 (1992), 62–76.
7. Wu Zhengchang, Norm of the Bernstein left quasi-interpolant operator, *J. Approx. Theory* **66** (1991), 36–43.

*Printed in Belgium*